J. Fluid Mech. (1999), *vol.* 387, *pp.* 343–352. Printed in the United Kingdom

A simple criterion for the sign of the pseudomomentum of modes in shallow water systems

By KEITA IGA†

Ocean Research Institute, University of Tokyo, Tokyo 164-8639, Japan

(Received 30 March 1998 and in revised form 3 December 1998)

A simple criterion is derived for determining the sign of the pseudomomentum of neutral modes in shallow water systems. The sign of the pseudomomentum is determined by the gradient of the dispersion curve on a wavenumber vs. phase-speed plane: a mode has pseudomomentum with the opposite sign to that of the gradient of the dispersion curve. In most cases, the sign of the pseudomomentum is also determined only from the value of its phase speed: the pseudomomentum of a mode is positive if its phase speed is faster than the velocity of the basic flow at any point and vice versa, but with a few exceptions.

1. Introduction

In fluid mechanics, including dynamic meteorology and physical oceanography, instability of parallel flows with various configurations has been extensively studied. A linear stability problem is often reduced to an eigenvalue problem. The procedure to solve an eigenvalue problem is established, and there is not much mathematical difficulty. However, physical interpretation is not easy. Since a subtle difference in the basic situation may cause a great difference in the stability, we need means to understand unstable modes in physical terms.

The concept of resonance between neutral waves is one of the ways to understand physically such complicated unstable modes (Cairns 1979; Hayashi & Young 1987). In particular, in the case of an instability caused by interaction between the layers in a two-layer situation, the unstable mode is clearly identified as follows: first, we obtain the eigenvalues for the reduced one-layer problem and then superpose the dispersion curves of the upper- and the lower-layer waves. Unstable modes in the two-layer problem are found where dispersion curves of the upper- and the lower-layer waves intersect, and they can be considered to be caused by resonance between the waves (Sakai 1989; Iga 1993, 1997).

Although an unstable mode may exist where a dispersion curve in the upper layer and one in the lower layer intersect, not all intersections cause unstable modes in the whole two-layer problem: an unstable mode appears in some cases and two modes interchange without causing instability in others. This difference is determined by the signs of the pseudomomenta of the two intersecting modes: an unstable mode appears if they are opposite, while no unstable mode appears if they are the same. Therefore

† Present address: Research Institute for Applied Mechanics, Kyushu University, Kasuga, Fukuoka 816-8580, Japan.

the sign of the pseudomomentum of a neutral wave is important information when investigating instability.

In cases where the basic flow in each layer is uniform, such as in Sakai (1989) and Iga (1993), we can presume the instability of the system without difficulty only from the configuration of the basic state, since it is easy to show that the pseudomomentum of a neutral wave has the same sign as that of intrinsic phase speed for uniform basic flow. However, when the basic flow in each layer is not uniform, such a simple criterion is not known. Thus, we want to derive simple criteria for determining the sign of the pseudomomentum which is valid also in general cases with non-uniform basic flow. Of course, it is hard to appreciate that the pseudomomentum of a mode in a basic flow with a little shear differs greatly from that in a uniform flow. When we solve a problem of frontal instability with a non-uniform basic flow, the sign of the pseudomomentum of a mode with a phase speed outside the range of the basic flow seems to be determined by whether the phase speed is faster or slower than the velocity of the basic flow, in the same way as in the case of uniform basic flow (Iga 1997). We want to investigate whether this is generally the case.

This paper is organized as follows. We will give the basic equations of the system considered and review the sign of the pseudomomentum in cases with a uniform basic flow in § 2. The main part of the paper is § 3: we will prove some theorems which give the sign of the pseudomomentum in general cases with non-uniform flow. A simple application of these theorems to stability of fronts will be shown in § 4.

2. Basic equations and pseudomomentum

We will consider a shallow water system over a rotating plane which is uniform in the x-direction. The basic flow is oriented in the x-direction and expressed as $U(y)$. For wave-type solutions with wavenumber k and phase speed c in the x-direction (proportional to $\mathrm{e}^{\mathrm{i}k(x-ct)}$), the basic linear equations are

$$-\mathrm{i}k(c-U)u = \left(f - \frac{\mathrm{d}U}{\mathrm{d}y}\right) v - \mathrm{i}kgh, \tag{2.1}$$

$$-\mathrm{i}k(c-U)v = -fu - g\frac{\mathrm{d}h}{\mathrm{d}y}, \tag{2.2}$$

$$-\mathrm{i}k(c-U)h = -\mathrm{i}kHu - \frac{\mathrm{d}}{\mathrm{d}y}(Hv). \tag{2.3}$$

From (2.1)–(2.3), the potential vorticity equation

$$-\mathrm{i}k(c-U)q = -\frac{\mathrm{d}Q}{\mathrm{d}y}v \tag{2.4}$$

is derived, where

$$q \equiv \frac{\zeta - Qh}{H}, \quad \zeta \equiv \mathrm{i}kv - \frac{\mathrm{d}u}{\mathrm{d}y}, \quad Q \equiv \frac{f - \mathrm{d}U/\mathrm{d}y}{H}$$

are the perturbation potential vorticity, the perturbation vorticity and the potential vorticity of the basic state, respectively.

In shallow water systems, there exists a conserved quality called the pseudomomentum†

$$M \equiv \frac{1}{4}\int \left(uh^* + hu^* - \frac{H^2|q|^2}{Q'} \right) \mathrm{d}y, \tag{2.5}$$

which is proportional to the square of the amplitude (see e.g. Hayashi & Young 1987; Sakai 1989). When the basic flow is uniform, the relation

$$(c-U)\int \left(uh^* + hu^* - \frac{H^2|q|^2}{Q'} \right) \mathrm{d}y = \int (H|u|^2 + H|v|^2 + g|h|^2)\mathrm{d}y \tag{2.6}$$

holds (Iga 1999), and the sign of the pseudomomentum M is the same as that of $(c-U)$, because the right-hand side of (2.6) is positive. Namely, the sign of the pseudomomentum is the same as that of the intrinsic phase speed.

3. Theorems on pseudomomentum

Even if the basic flow is not uniform, the following theorems, which give the sign of the pseudomomentum M, generally hold.

Lemma 1. *The relation* $(\mathrm{d}c/\mathrm{d}k)M = -(1/2k)\int H|v|^2\mathrm{d}y$ *holds for a neutral mode.*

Proof. $(2.1)^{(1)} \times h^{(2)*}$ leads to

$$-\mathrm{i}k^{(1)}(c^{(1)}-U)u^{(1)}h^{(2)*} = QHv^{(1)}h^{(2)*} - \mathrm{i}k^{(1)}gh^{(1)}h^{(2)*}, \tag{3.1}$$

and $(2.3)^{(1)} \times u^{(2)*}$ to

$$-\mathrm{i}k^{(1)}(c^{(1)}-U)h^{(1)}u^{(2)*} = -\mathrm{i}k^{(1)}Hu^{(1)}u^{(2)*} - u^{(2)*}\frac{\mathrm{d}}{\mathrm{d}y}(Hv^{(1)}). \tag{3.2}$$

Since (3.1)+(3.2) becomes

$$\begin{aligned}&-\mathrm{i}k^{(1)}(c^{(1)}-U)(u^{(1)}h^{(2)*} + h^{(1)}u^{(2)*})\\ &\quad = -\mathrm{i}k^{(1)}gh^{(1)}h^{(2)*} - \mathrm{i}k^{(1)}Hu^{(1)}u^{(2)*} - \mathrm{i}k^{(2)}Hv^{(1)}v^{(2)*} - H^2v^{(1)}q^{(2)*} - \frac{\mathrm{d}}{\mathrm{d}y}(Hv^{(1)}u^{(2)*}),\end{aligned}$$

modification using (2.4) and integration over the region leads to

$$\begin{aligned}&\int (c^{(1)}-U)\left(u^{(1)}h^{(2)*} + h^{(1)}u^{(2)*} - \frac{H^2q^{(1)}q^{(2)*}}{Q'} \right)\mathrm{d}y\\ &\qquad = \int \left(Hu^{(1)}u^{(2)*} + gh^{(1)}h^{(2)*} + \frac{k^{(2)}}{k^{(1)}}Hv^{(1)}v^{(2)*} \right)\mathrm{d}y.\end{aligned} \tag{3.3}$$

Replacing $^{(1)}$ and $^{(2)}$ and taking the complex conjugate of (3.3), we obtain

† When Q' vanishes, q is also zero for a non-singular mode, as is seen from (2.4). Thus, (2.5) does not diverge even if there are points which satisfy $Q'=0$. This is also evident from the expression used in Hayashi & Young (1987) and Sakai (1989) $M \equiv (1/4)\int(uh^* + hu^* - Q'H^2|\eta|^2)\mathrm{d}y$, where η is displacement in the y-direction and thus $\eta = v/[-\mathrm{i}k(c-U)]$. Since Q' in a denominator is always accompanied by q in the numerator, the following discussions hold even if Q' vanishes, as far as non-singular modes are concerned.

$$\int (c^{(2)*} - U) \left(u^{(1)}h^{(2)*} + h^{(1)}u^{(2)*} - \frac{H^2 q^{(1)} q^{(2)*}}{Q'} \right) \mathrm{d}y = \int \left(Hu^{(1)}u^{(2)*} + gh^{(1)}h^{(2)*} + \frac{k^{(1)}}{k^{(2)}} Hv^{(1)}v^{(2)*} \right) \mathrm{d}y, \quad (3.4)$$

and (3.3)–(3.4) leads to

$$(c^{(1)} - c^{(2)*}) \int \left(u^{(1)}h^{(2)*} + h^{(1)}u^{(2)*} - \frac{H^2 q^{(1)} q^{(2)*}}{Q'} \right) \mathrm{d}y = \left(\frac{k^{(2)}}{k^{(1)}} - \frac{k^{(1)}}{k^{(2)}} \right) \int Hv^{(1)}v^{(2)*} \mathrm{d}y. \quad (3.5)$$

Since c is real and thus $c^{(2)*} = c^{(2)}$ for a neutral wave, dividing both sides of (3.5) by $k^{(1)} - k^{(2)}$ and letting $k^{(2)} \to k^{(1)}$, we get

$$\frac{\mathrm{d}c}{\mathrm{d}k} \int \left(uh^* + hu^* - \frac{H^2 |q|^2}{Q'} \right) \mathrm{d}y = -\frac{2}{k} \int H|v|^2 \mathrm{d}y, \quad (3.6)$$

which shows the required relation.

In the special case of $Q \equiv 0$, it is already known that this relation holds (Hayashi & Young 1987).

LEMMA 2. *If a mode satisfies* $v \equiv 0$ *for a certain* k, *then* $\mathrm{d}c/\mathrm{d}k = 0$ *for all* k.

Proof. Putting $v \equiv 0$ in the basic equations (2.1)–(2.3), we get

$$(c - U)u = gh, \quad (3.7)$$

$$0 = -fu - g\frac{\mathrm{d}h}{\mathrm{d}y}, \quad (3.8)$$

$$(c - U)h = Hu, \quad (3.9)$$

which do not include k. Therefore, if there exist an eigenvalue and a corresponding eigenfunction which satisfy (3.7)–(3.9) for a certain value of k, they are also an eigenvalue and a corresponding eigenfunction for any value of k. Thus, the eigenvalue c does not change even if k changes; this means $\mathrm{d}c/\mathrm{d}k = 0$ for all k.

THEOREM 1. *If* $\mathrm{d}c/\mathrm{d}k \gtrless 0$, *then* $M \lessgtr 0$.

Proof. From Lemma 1, if $\mathrm{d}c/\mathrm{d}k > 0$, then $M \leqslant 0$ holds, since the right-hand side of the equation in Lemma 1 satisfies $-(1/k)\int H|v|^2\mathrm{d}y \leqslant 0$. The equality holds only when $-(1/k)\int H|v|^2\mathrm{d}y = 0$ (i.e. $v \equiv 0$), but in this case, $\mathrm{d}c/\mathrm{d}k = 0$ from Lemma 2, which leads to a contradiction to the assumption. Hence, there is no choice but $M < 0$. In the same way, if $\mathrm{d}c/\mathrm{d}k < 0$, then $M > 0$ holds.

LEMMA 3. *If* $\mathrm{d}c/\mathrm{d}k = 0$ *and* $(c - U_{max}) > 0$, *then* $M > 0$, *and if* $\mathrm{d}c/\mathrm{d}k = 0$ *and* $(c - U_{min}) < 0$, *then* $M < 0$.

Proof. From Lemma 1, $\mathrm{d}c/\mathrm{d}k = 0$ leads to $-(1/k)\int H|v|^2\mathrm{d}y = 0$, which means $v \equiv 0$. From (2.4), $v \equiv 0$ leads to $q \equiv 0$. Moreover, using the relation (3.7), we can express M as

$$M = \frac{1}{2} \int \frac{(c - U)|u|^2}{g} \mathrm{d}y,$$

in this case. If $(c - U_{max}) > 0$, then the integrand is always positive and thus $M > 0$. In the same way, if $(c - U_{min}) < 0$, then $M < 0$.

LEMMA 4. *For a mode with a phase speed outside the range of the basic flow U, $\mathrm{d}c/\mathrm{d}k$ does not change its sign, even if k changes.*

Proof. If we assume that the sign of the gradient of dispersion curve changes from $\mathrm{d}c/\mathrm{d}k > 0$ to $\mathrm{d}c/\mathrm{d}k < 0$, $\mathrm{d}c/\mathrm{d}k$ must vanish at a certain k, but this means that $v \equiv 0$ from Lemma 1, and that c is constant for this mode from Lemma 2, which is contradictory to the assumption.

LEMMA 5. *If $(c - U_{max}) > |f|_{max}/2k$, then $\mathrm{d}c/\mathrm{d}k \leqslant 0$, and if $(c - U_{min}) < -|f|_{max}/2k$, then $\mathrm{d}c/\mathrm{d}k \geqslant 0$.*

Proof. Substitution of (2.1) into (2.2) $\times\, [-\mathrm{i}k(c - U)]$ eliminates the variable u, and we get

$$-k^2(c-U)^2 v = -fHQv + \mathrm{i}kfgh + \mathrm{i}k(c-U)g\frac{\mathrm{d}h}{\mathrm{d}y}. \tag{3.10}$$

In the same way, substitution of (2.1) into (2.3) $\times\, [-\mathrm{i}k(c - U)]$ eliminates u, and we get

$$-k^2(c-U)^2 h = -\mathrm{i}kH^2Qv - k^2gHh + \mathrm{i}k(c-U)\frac{\mathrm{d}}{\mathrm{d}y}(Hv), \tag{3.11}$$

while $(3.10)^{(1)} \times v^{(2)*}$ leads to

$$\begin{aligned} &- k^{(1)^2}(c^{(1)} - U)^2 v^{(1)} v^{(2)*} \\ &\quad = -fHQv^{(1)}v^{(2)*} + \mathrm{i}k^{(1)}fgh^{(1)}v^{(2)*} + \mathrm{i}k^{(1)}(c^{(1)} - U)gv^{(2)*}\frac{\mathrm{d}h^{(1)}}{\mathrm{d}y}, \end{aligned} \tag{3.12}$$

and $(3.11)^{(2)*} \times h^{(1)}$ to

$$\begin{aligned} &-k^{(2)^2}(c^{(2)*} - U)^2 h^{(1)} h^{(2)*} \\ &\quad = \mathrm{i}k^{(2)}H^2Qh^{(1)}v^{(2)*} - k^{(2)^2}gHh^{(1)}h^{(2)*} - \mathrm{i}k^{(2)}(c^{(2)*} - U)h^{(1)}\frac{\mathrm{d}}{\mathrm{d}y}(Hv^{(2)*}). \end{aligned} \tag{3.13}$$

$(3.12) \times H/k^{(1)}k^{(2)}(c^{(1)} - U)(c^{(2)*} - U) - (3.13) \times g/k^{(2)^2}(c^{(2)*} - U)^2$ becomes

$$\begin{aligned} &-\frac{k^{(1)}}{k^{(2)}}\frac{(c^{(1)} - U)}{(c^{(2)*} - U)}Hv^{(1)}v^{(2)*} + gh^{(1)}h^{(2)*} \\ &\quad = -\frac{fH^2Qv^{(1)}v^{(2)*}}{k^{(1)}k^{(2)}(c^{(1)} - U)(c^{(2)*} - U)} + \frac{\mathrm{i}fgHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(1)} - U)(c^{(2)*} - U)} + \frac{\mathrm{i}gHv^{(2)*}}{k^{(2)}(c^{(2)*} - U)}\frac{\mathrm{d}h^{(1)}}{\mathrm{d}y} \\ &\qquad -\frac{\mathrm{i}gH^2Qh^{(1)}v^{(2)*}}{k^{(2)}(c^{(2)*} - U)^2} + \frac{g^2Hh^{(1)}h^{(2)*}}{(c^{(2)*} - U)^2} + \frac{\mathrm{i}gh^{(1)}}{k^{(2)}(c^{(2)*} - U)}\frac{\mathrm{d}}{\mathrm{d}y}(Hv^{(2)*}) \\ &\quad = -\frac{fH^2Qv^{(1)}v^{(2)*}}{k^{(1)}k^{(2)}(c^{(1)} - U)(c^{(2)*} - U)} + \frac{\mathrm{i}fgHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(1)} - U)(c^{(2)*} - U)} \\ &\qquad -\frac{\mathrm{i}gfHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(2)*} - U)^2} + \frac{g^2Hh^{(1)}h^{(2)*}}{(c^{(2)*} - U)^2} + \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{i}gh^{(1)}}{k^{(2)}(c^{(2)*} - U)}Hv^{(2)*}\right). \end{aligned} \tag{3.14}$$

Integrating (3.14), we obtain

$$\int \left(-\frac{k^{(1)}}{k^{(2)}}\frac{(c^{(1)}-U)}{(c^{(2)*}-U)}Hv^{(1)}v^{(2)*}+gh^{(1)}h^{(2)*}\right)\mathrm{d}y$$
$$=\int\left(-\frac{fH^2Qv^{(1)}v^{(2)*}}{k^{(1)}k^{(2)}(c^{(1)}-U)(c^{(2)*}-U)}+\frac{\mathrm{i}fgHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(1)}-U)(c^{(2)*}-U)}\right.$$
$$\left.-\frac{\mathrm{i}gfHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(2)*}-U)^2}+\frac{g^2Hh^{(1)}h^{(2)*}}{(c^{(2)*}-U)^2}\right)\mathrm{d}y. \tag{3.15}$$

Replacing $^{(1)}$ and $^{(2)}$, taking the complex conjugate of (3.15) and subtracting it from the original (3.15), we obtain

$$\int -\left(\frac{k^{(1)}}{k^{(2)}}\frac{(c^{(1)}-U)}{(c^{(2)*}-U)}-\frac{k^{(2)}}{k^{(1)}}\frac{(c^{(2)*}-U)}{(c^{(1)}-U)}\right)Hv^{(1)}v^{(2)*}\mathrm{d}y$$
$$=\int\left[\left(\frac{1}{(c^{(1)}-U)}-\frac{1}{(c^{(2)*}-U)}\right)\frac{\mathrm{i}fgHh^{(1)}v^{(2)*}}{k^{(2)}(c^{(2)*}-U)}\right.$$
$$+\left(\frac{1}{(c^{(2)*}-U)}-\frac{1}{(c^{(1)}-U)}\right)\frac{\mathrm{i}fgHh^{(2)*}v^{(1)}}{k^{(1)}(c^{(1)}-U)}$$
$$\left.+\left(\frac{1}{(c^{(2)*}-U)^2}-\frac{1}{(c^{(1)}-U)^2}\right)g^2Hh^{(1)}h^{(2)*}\right]\mathrm{d}y.$$

Since c is real and thus $c^{(2)*}=c^{(2)}$ for a neutral wave, dividing both sides of (3.5) by $k^{(1)}-k^{(2)}$ and letting $k^{(2)}\to k^{(1)}$, we get

$$-\int H|v|^2\mathrm{d}y=\frac{\mathrm{d}c}{\mathrm{d}k}\int\left[\frac{kH|v|^2}{c-U}+\frac{kg^2H|h|^2}{(c-U)^3}-\frac{\mathrm{i}fgH}{2(c-U)^3}(hv^*-h^*v)\right]\mathrm{d}y. \tag{3.16}$$

The integrand of the right-hand side of (3.16) is modified to

$$\frac{kg^2H}{(c-U)^3}\left|h+\frac{\mathrm{i}fv}{2kg}\right|^2+\frac{H}{k(c-U)^3}\left[k^2(c-U)^2-\left(\frac{f}{2}\right)^2\right]|v|^2,$$

which proves the lemma.

From this lemma, we know the sign of the pseudomomentum for the modes in the region indicated in figure 1. In a non-rotating situation ($f\equiv 0$) as a special case, if $(c-U_{max})>0$, then $M>0$, and if $(c-U_{min})<0$ then $M<0$ as is evident from this lemma.

THEOREM 2. *If* $(c-U_{max})>0$, *then* $M>0$, *and if* $(c-U_{min})<0$, *then* $M<0$, *for a neutral mode which is not connected to an unstable mode at a larger wavenumber* k.

Proof. First, let us consider the case of $(c-U_{max})>0$. If we assume $\mathrm{d}c/\mathrm{d}k>0$ for a mode which is not connected to an unstable mode at larger k, even if the wavenumber change $\mathrm{d}c/\mathrm{d}k>0$ would still hold, owing to Lemma 4; $\mathrm{d}c/\mathrm{d}k>0$ would lead to $(c-U_{max})>|f|_{max}/2k$ at a certain large enough wavenumber. However, $\mathrm{d}c/\mathrm{d}k$ must be negative in this region from Lemma 5, which is a contradiction. Therefore, for a mode with $(c-U_{max})>0$ (and not connected to an unstable mode at larger k), $\mathrm{d}c/\mathrm{d}k\leqslant 0$ must hold. The inequality $M>0$ holds from Theorem 1 if $\mathrm{d}c/\mathrm{d}k<0$, and from Lemma 3 if $\mathrm{d}c/\mathrm{d}k=0$. In the same way, $M<0$ holds, when $(c-U_{min})<0$.

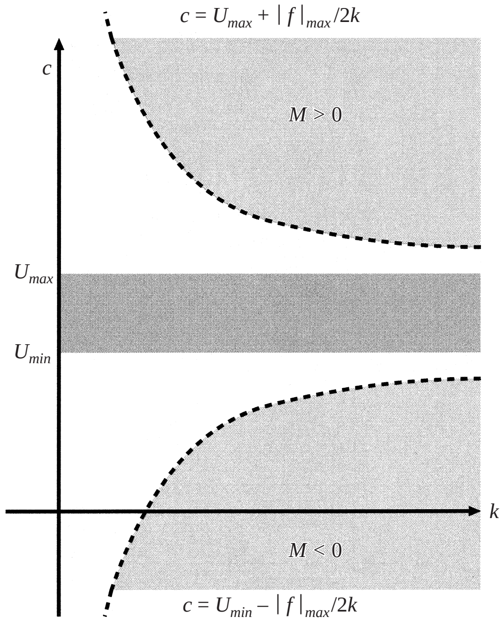

FIGURE 1. Region where the sign of pseudomomentum is determined from Lemma 5. The darker-shaded horizontal region indicates where the velocity of the basic flow exists. It is known from Lemma 5 that modes in the lighter-shaded region indicated as $M > 0$ have positive pseudomomentum and that modes in the region indicated as $M < 0$ have negative pseudomomentum.

4. Discussion

From Theorem 2, we can roughly conclude that pseudomomentum usually has the same sign as that of the intrinsic phase speed, even if the basic flow is not uniform. Moreover, even if the sign of the pseudomomentum cannot be determined from the criterion of Theorem 2 (e.g. modes whose phase speed is inside the range of the basic flow velocity), it is determined by the gradient of the dispersion curve from Theorem 1. Numerical results of Iga (1997) are consistent with these criteria. Although the sign of the pseudomomentum of a wave is the same as that of the intrinsic phase speed in most cases, exceptional cases may exist according to Theorem 2: a mode which is connected to an unstable mode at a larger wavenumber may have a negative (positive) pseudomomentum even if its intrinsic phase speed is positive (negative) (figure 2), and such an exception has been found in reality (Yamasaki & Wada 1972).†

Since the discussion in this paper is based on the assumption of regularity of eigenfunctions, we can apply the theorems only to non-singular modes: for continuous modes which have singularities inside their eigenfunctions, we can derive no information about the sign of the pseudomomentum from these theorems. It is already known, however, that the sign of the pseudomomentum of a continuous mode is decided by the sign of the potential vorticity gradient at the critical level: the pseudomomentum is negative (positive) when the potential vorticity gradient is positive (negative) (Iga

† Since Yamasaki & Wada (1972) did not treat a shallow water system but a two-dimensional barotropic system, it is outside the applicability condition of this theorem. However, when we make the depth of a shallow water system infinitely large, the phase speeds of some modes remain finite, while those of others diverge to infinity. Since the former in the limiting case correspond to modes in a two-dimensional barotropic system, the same theorems must be applicable also to two-dimensional barotropic systems.

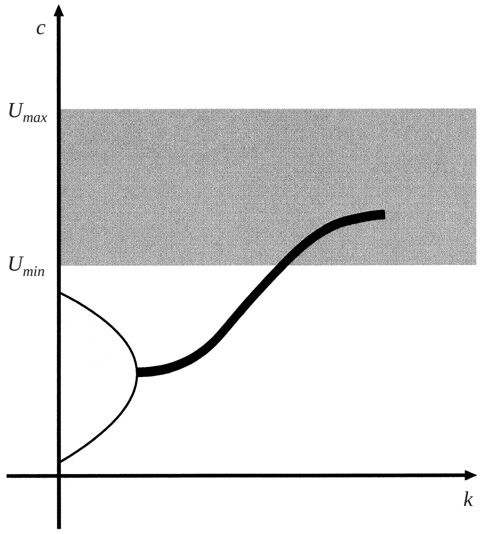

FIGURE 2. A neutral mode which is connected to an unstable mode at a larger wavenumber. The thin line indicates a dispersion curve of a neutral mode, and the thick line that of an unstable mode. The neutral mode has a negative gradient of the dispersion curve and therefore has a positive pseudomomentum, although its phase speed is smaller than U_{min}.

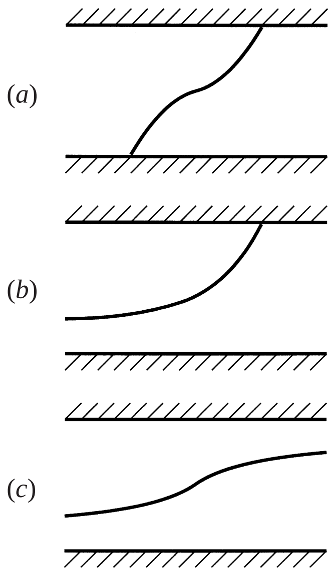

FIGURE 3. Configuration of frontal models: (*a*) the frontal surface intersects both the upper and lower boundaries; (*b*) the frontal surface intersects only the upper boundary; (*c*) the frontal surface intersects neither boundary. The situation of (*c*) is not usually called a 'front', and thus we do not treat this situation here.

1999). Therefore, combining this knowledge and the theorems derived here, we can easily judge the sign of the pseudomomentum, whether the mode is non-singular or continuous.

Let us apply these theorems to the instability of fronts. While many frontal models investigated previously are unstable (e.g. Killworth, Paldor & Stone 1984; Iga 1993, 1997), we can show that the two-layer frontal model is almost always unstable, by

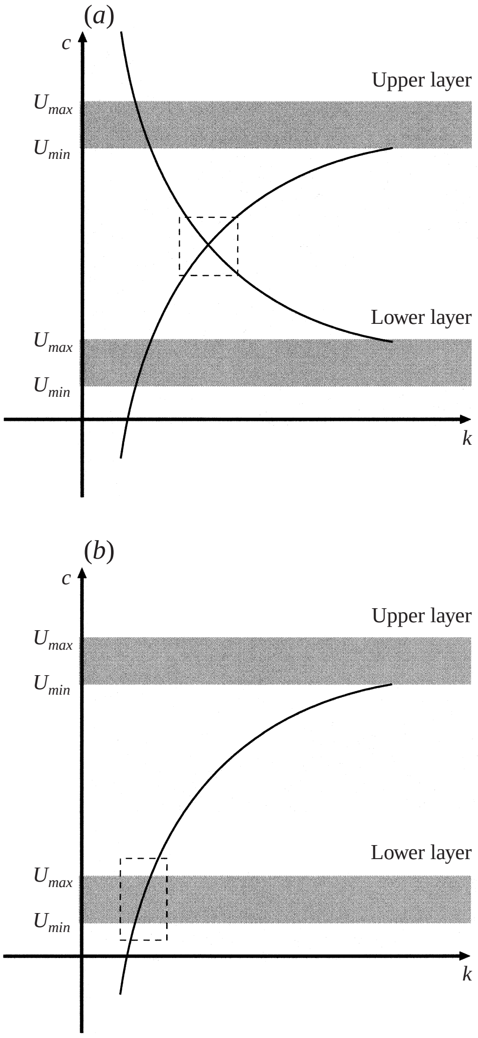

FIGURE 4. The range of velocity of the basic flow in both layers and dispersion curves of Poincaré (gravity) modes. There must be instability in the dashed square. (*a*) The dispersion curves of Poincaré modes in both layers intersect. Since the signs of their pseudomomenta are opposite, from Theorem 2, this intersection causes instability. (*b*) The dispersion curve of a Poincaré mode enters the velocity range of the basic flow in the lower layer. Instability occurs, since this mode has a negative pseudomomentum from Theorem 2 and there is a point where $Q' < 0$ in the lower layer.

applying these theorems. A 'front' is usually considered to be where the frontal surface intersects either the ground (sea bottom) or the tropopause (sea surface). Hence, we do not consider the situation of figure 3(*c*) here; we only consider the situations of figure 3(*a, b*), where the frontal surface intersects the upper boundary and/or the lower one.

Note that in a shallow water system, there exist Poincaré (gravity) modes for which $|c| \to \infty$ as $k \to 0$ and $c \to U(y_0) \pm (gH(y_0))^{1/2}$ as $k \to \infty$ (where y_0 is the point where $H(y)$ becomes minimum). Then, in the situation of figure 3(*a*), where $H_{min} = 0$ for both layers, the dispersion curves of Poincaré modes in both layers must cross. Since the signs of the pseudomomenta of these Poincaré modes are opposite from Theorem 2, instability occurs around this crosspoint (figure 4*a*).

On the other hand, in the situation of figure 3(b), there may be a possibility that dispersion curves of Poincaré modes in both layers do not intersect, since the minimum depth of the lower layer is finite. Nevertheless, in this case, instability usually occurs where the dispersion curve of a Poincaré mode in the upper layer goes through the velocity region of basic flow in the lower layer (figure 4b) for the following reason. The absolute vorticity in the lower layer approaches f as $y \to \pm\infty$, if the flow becomes uniform. The potential vorticity is larger in the negative direction, since the depth of the layer is smaller. Thus, there must be a point where the potential vorticity gradient is negative. An unstable mode is necessarily formed, when a dispersion curve of a mode with negative pseudomomentum enters a region of basic flow with negative potential vorticity gradient (Iga 1999).

5. Conclusions

We have derived some simple criteria by which to judge the sign of the pseudomomentum of a neutral mode in shallow water systems, which is important in the concept of resonance between waves. The sign of the pseudomomentum is determined by the gradient of dispersion curves on the wavenumber vs. phase-speed plane: the pseudomomentum is negative if the gradient of the dispersion curve is positive, while the pseudomomentum is positive if the gradient is negative.

Moreover, the sign of the pseudomomentum is usually known only from the value of its phase speed: the pseudomomentum is positive for a mode whose phase speed is faster than the velocity of the basic flow at any point, and is negative for a mode whose phase speed is slower than the velocity of the basic flow at any point. However, we must note that there is an exception to this criterion: the pseudomomentum of a mode which is connected to an unstable mode at a larger wavenumber may not have the sign expected from this criterion.

From these theorems and knowledge of the condition for a non-singular mode and the basic flow to interact, we can conclude that the system called a 'front' is almost always unstable.

The author thanks Professors R. Kimura and H. Niino for their useful comments. The author was supported by JSPS Research Fellowships for Young Scientists, and this work was supported in part by the Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture of Japan.

REFERENCES

Cairns, R. A. 1979 The role of negative energy waves in some instabilities of parallel flows. *J. Fluid Mech.* **92**, 1–14.

Hayashi, Y.-Y. & Young, W. R. 1987 Stable and unstable shear modes on rotating parallel flows in shallow water. *J. Fluid Mech.* **184**, 477–504.

Iga, K. 1993 Reconsideration of Orlanski's instability theory of frontal waves. *J. Fluid Mech.* **255**, 213–236.

Iga, K. 1997 Instability of front with a layer of uniform potential vorticity. *J. Met. Soc. Japan* **75**, 1–11.

Iga, K. 1999 Critical layer instability as a resonance between a non-singular mode and continuous modes. *Fluid Dyn. Res.* in press.

Killworth, P. D., Paldor, N. & Stern, M. E. 1984 Wave propagation and growth on a surface front in a two-layer geostrophic current. *J. Mar. Res.* **42**, 761–785.

Sakai, S. 1989 Rossby–Kelvin instability: a new type of ageostrophic instability caused by a resonance between Rossby waves and gravity waves. *J. Fluid Mech.* **202**, 149–176.

Yamasaki, M. & Wada, M. 1972 Barotropic instability of an easterly zonal current. *J. Met. Soc. Japan* **50**, 110–121.